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Optimization of plastic conical shells loaded by a rigid central boss

J. Lellep*, E. Puman

Institute of Applied Mathematics, Tartu University, 51014 Tartu, Estonia

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Abstract

Simply supported conical shells loaded by the central rigid boss with vertical load are considered. The thickness of the shell wall is assumed to be piece-wise constant with a finite number of jumps. The minimum weight design of the shell is established under the condition that the load carrying capacity of the shell is given. The shell material is assumed to be an ideal rigid-plastic one obeying the generalized diamond yield condition and associated flow law. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Conical shells have many attractive applications in engineering. Thin-walled conical shells are used not only as nozzles for pressure vessels but as the substantial parts of different machines and structures in mining, agriculture and elsewhere.

Limit analysis of conical shells subjected to lateral distributed loading has been investigated by Hodge (1963), Kuech and Lee (1965) and others in the case of a Tresca material. The shells subjected to uniformly distributed normal pressure and edge tension were considered by Hodge and Lakshmikantham (1963).

The collapse loads for rigid-plastic conical shells loaded by a central boss have been defined under different assumptions by Hodge (1963), Onat (1960), Lance and Onat (1963), Lance and Lee (1969) and Hodge and Deruntz (1964) assuming the material obeys Tresca's yield condition.

In the present paper, following the variational approach by Lellep (1991), a minimum weight design

* Corresponding author. Fax: +372-375-862.
E-mail address: appliedm@ut.ee (J. Lellep).

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Nomenclature

P	vertical load
a, R	radii of the shell
h	thickness of the shell
h_j, a_j	design parameters
r	current radius
V	material volume of the shell
N_1, N_2	membrane forces
M_1, M_2	bending moments
$\dot{U}$, $\dot{W}$	displacement rates
$\dot{\epsilon}_1$, $\dot{\epsilon}_2$	strain rates
$\dot{\kappa}_1$, $\dot{\kappa}_2$	curvature rates
φ	angle of the slope
α_j, γ_j	dimensionless parameters
u, w	non-dimensional displacement rates
$n_{1,2}$, $m_{1,2}$	non-dimensional stress resultants
ϱ, α	non-dimensional radii
q	non-dimensional load parameter
k	geometrical parameter
M_*, N_*	limit moment and limit force
h_*	thickness of the reference shell
σ_0	yield stress
v	non-dimensional volume
q_0	load carrying capacity
n	the number of jumps in the thickness
λ_j	Lagrangian multipliers
V_*	material volume of the reference shell
e	coefficient of economy
D_j	an interval

technique is developed for conical shells loaded by the central rigid boss. The shell wall is assumed to be of piece-wise constant thickness.

In the case of a shell made of an elastic material, any sudden discontinuity of wall thickness will result in some form of stress concentration. However, in this paper, it is assumed that the material is an ideal rigid-plastic material. As was shown in early works by Sheu and Prager (1969) and also by Lamblin et al. (1985), when considering axisymmetric plates of piece-wise constant thickness, no stress concentration will occur in the case of pure plastic deformations. Therefore, the phenomenon of stress concentration will not be considered in the current paper.

The material of the shell is assumed to obey the Tresca yield condition and associated flow rule.

The yield surface in the space of generalized stresses corresponding to the original Tresca condition is presented in the form of generalized diamond yield conditions suggested by Jones and Ich (1972).

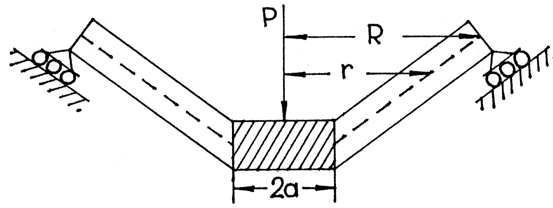

Fig. 1. Conical shell loaded by the central boss.

2. Formulation of the problem and basic equations

Let us consider a conical shell simply supported at the outer edge of radius R and loaded by the central boss with vertical load P (Fig. 1). It is assumed that the central boss with radius $a > R/2$ is absolutely rigid and the shell is clamped to the boss at the inner edge of the shell with radius a.

Assume that the thickness of the shell is piece-wise constant, e.g.,

$$h = h_j,\ r \in (a_j, a_{j+1}), \tag{1}$$

where $j = 0,\ldots, n$ and $a_0 = a$, $a_{n+1} = R$. Here, the quantities h_j ($j = 0,\ldots, n$) and a_j ($j = 1,\ldots, n$) are considered as unknown constants. We are looking for the design of the shell for which the weight (or material volume) attains the minimal value for given load carrying capacity. The material volume of the shell wall with a given thickness Eq. (1) can be presented as

$$V = \sum_{j=0}^{n} h_j \left(a_{j+1}^2 - a_j^2 \right) \cdot \frac{\pi}{\cos\varphi}. \tag{2}$$

Here, φ stands for the angle of inclination of a generator of the shell.

The equilibrium equations for a shell element have the form (see Hodge, 1963)

$$\frac{\mathrm{d}}{\mathrm{d}r}(rN_1) - N_2 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(rM_1) - M_2 - rN_1 \frac{\sin\varphi}{\cos^2\varphi} + \frac{P}{2\pi\cos^2\varphi} = 0, \tag{3}$$

where N_1, N_2 are the membrane forces and M_1, M_2 the principal moments, respectively.

Let $\dot{W}$ and $\dot{U}$ be the displacement rates in the transversal and tangential directions, respectively. Strain and curvature rates for conical shells can be presented as

$$\dot{\epsilon}_1 = \frac{\mathrm{d}\dot{U}}{\mathrm{d}r}\cos\varphi,$$

$$\dot{\kappa}_1 = -\frac{h}{4}\frac{\mathrm{d}^2\dot{W}}{\mathrm{d}r^2} \cdot \cos^2\varphi,$$

$$\dot{\epsilon}_2 = \frac{1}{r}(\dot{U}\cos\varphi + \dot{W}\sin\varphi)$$

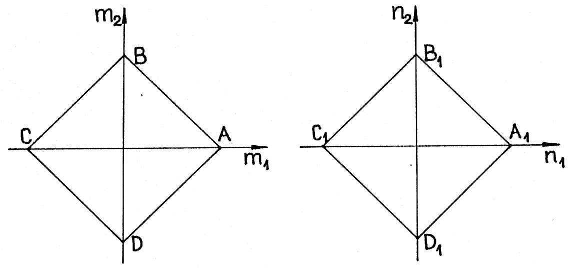

Fig. 2. Generalized diamond yield condition.

$$\dot{\kappa}_2 = -\frac{h}{4r}\frac{\mathrm{d}\dot{W}}{\mathrm{d}r}\cdot\cos^2\varphi. \tag{4}$$

The material of the shell is assumed to be an ideal rigid-plastic one obeying Tresca's yield condition and associated flow law. The yield surface in the four-dimensional space of stress resultants is of quite complex form, even in the case of Tresca's yield condition. A series of different simplifications of the exact yield surface have been developed by Hodge (1963). It was shown by Jones and Ich (1972) that the generalized diamond yield condition leads to predictions of the load carrying capacity of axisymmetric shells which are close to the exact results. The generalized diamond yield condition will be used in the present paper (Fig. 2).

It appears to be convenient to carry out the analysis in terms of dimensionless quantities defined by

$$\varrho = \frac{r}{R},$$

$$\alpha_j = \frac{a_j}{R},$$

$$\gamma_j = \frac{h_j}{h_*},$$

$$w = \frac{W}{R},$$

$$u = \frac{U}{R},$$

$$n_{1,2} = \frac{N_{1,2}}{N_*},$$

$$m_{1,2} = \frac{M_{1,2}}{M_*},$$

$$k = \frac{M_*}{RN_*}\cdot\frac{\cos^2\varphi}{\sin^2\varphi},$$

$$q = \frac{P}{2\pi R N_* \sin\varphi} - \frac{M_* \cos^2\varphi}{R N_* \sin\varphi}$$

$$\alpha = \frac{a}{R}. \tag{5}$$

Here, M_* and N_* stand for the limit moment and limit force, respectively, for a shell of thickness h_*. Thus, $M_* = \sigma_0 h_*^2/4$, $N_* = \sigma_0 h_*$, σ_0 being the yield stress of the shell material.

Making use of the dimensionless variables Eq. (5), the cost criterion Eq. (2) can be presented as

$$v = \sum_{j=0}^{n} \gamma_j \left(\alpha_{j+1}^2 - \alpha_j^2\right), \tag{6}$$

whereas the equilibrium equations Eq. (3) take the form

$$(\varrho n_1)' - n_2 = 0$$

$$k\left[(\varrho m_1)' - m_2 + 1\right] - \varrho n_1 + q = 0. \tag{7}$$

Here, primes denote the differentiation with respect to ϱ and $v = V \cos\varphi/(\pi h_* R^2)$.

The statical boundary conditions are

$$m_1(\alpha) = \gamma_0^2,$$

$$m_1(1) = 0$$

$$n_1(1) = 0, \tag{8}$$

since a hinge circle is assumed to be located at the internal edge of the shell.

3. The shell of constant thickness

Consider first the case when $h = h_*$, where $h_* = \text{const}$. Assumptions about statical admissibility of the stress field lead to the stress regime, AB and D_1A_1 (Fig. 2). Prompted by this yield regime, one can state that

$$m_2 = 1 - m_1$$

$$n_2 = -1 + n_1. \tag{9}$$

Substituting Eq. (9) in Eq. (7) after integration and taking Eq. (8) into account, one has

$$n_1 = -\ln\varrho$$

$$m_1 = \frac{q}{2k}\left(\frac{1}{\varrho^2} - 1\right) - \frac{\varrho \ln\varrho}{3k} + \frac{1}{9k}\left(\varrho - \frac{1}{\varrho^2}\right). \tag{10}$$

Taking $m_1(\alpha) = 1$ in Eq. (10) leads to the limit load,

$$q_0 = \frac{2}{9(1-\alpha^2)}(1 + 9k\alpha^2 + \alpha^3(3\ln\alpha - 1)). \tag{11}$$

It can be seen from Fig. 2 that the associated flow law states that

$$\dot{\epsilon}_1 = -\dot{\epsilon}_2$$

$$\dot{\kappa}_1 = \dot{\kappa}_2. \tag{12}$$

It immediately follows from Eqs. (12) and (3) that

$$\dot{w} = c\ln\varrho$$

$$\dot{u} = c\tan\varphi\left(1 - \ln\varrho + \frac{\alpha}{\varrho}(\ln\alpha - 1)\right), \tag{13}$$

where the boundary conditions $\dot{w}(1) = 0$ and $\dot{u}(\alpha) = 0$ are taken into account.

4. Conical shell of piece-wise constant thickness

It might be expected that the yield regime, $AB{-}D_1A_1$, takes place in each segment of the shell as in the previous case. Let $\gamma = \gamma_j$ for $\varrho \in D_j$, where $D_j = (\alpha_j, \alpha_{j+1})$ for $j = 0, \ldots, n$. Thus,

$$n_2 = n_1 - \gamma_j \tag{14}$$

and

$$m_2 = \gamma_j^2 - m_1 \tag{15}$$

for $\varrho \in D_j$ ($j = 0, \ldots, n$).

Eqs. (14) and (7) easily give

$$n_1 = -\gamma_j \ln\varrho + c_j \tag{16}$$

for $\varrho \in D_j$ ($j = 0, \ldots, n$). Arbitrary constants, c_j, can be defined making use of Eq. (8) and continuity of the membrane force n_1 at $\varrho = \alpha_j$ ($j = 1, \ldots, n$). This leads to the relations

$$n_1 = -\gamma_j \ln\varrho + \sum_{i=j+1}^{n}(\gamma_{i-1} - \gamma_i)\ln\alpha_i \tag{17}$$

for $\varrho \in D_j$ $j = 0, \ldots, n-1$ and

$$n_1 = -\gamma_n \ln\varrho \tag{18}$$

for $\varrho \in D_n$.

Inserting Eqs. (15), (17) and (18) in the second equation in Eq. (7), one obtains

$$m_1' + \frac{2}{\varrho}m_1 = \frac{1}{\varrho}\left(\gamma_j^2 - 1 - \frac{q}{k}\right) + \frac{1}{k}\left[-\gamma_j\ln\varrho + \sum_{i=j+1}^{n}(\gamma_{i-1} - \gamma_i)\ln\alpha_i\right] \tag{19}$$

for $\varrho \in D_j$ ($j = 0,..., n-1$) and

$$m_1' + \frac{2}{\varrho} m_1 = \frac{1}{\varrho}\left(\gamma_n^2 - 1 - \frac{q}{k}\right) - \frac{\gamma_n}{k} \ln \varrho \tag{20}$$

for $\varrho \in D_n$.

Integrating Eqs. (19) and (20) leads to the bending moment distribution:

$$m_1 = \frac{1}{2}\left(\gamma_j^2 - 1 - \frac{q}{k}\right) - \frac{\varrho \gamma_j}{9k}(3 \ln \varrho - 1) + \frac{c_j}{\varrho^2} + \frac{\varrho}{3k}\sum_{i=j+1}^{n}(\gamma_{i-1} - \gamma_i)\ln \alpha_i \tag{21}$$

for $\varrho \in D_j$ ($j = 0,..., n-1$) and

$$m_1 = \frac{1}{2}\left(\gamma_n^2 - 1 - \frac{q}{k}\right) - \frac{\gamma_n \varrho}{9k}(3 \ln \varrho - 1) + \frac{c_n}{\varrho^2} \tag{22}$$

for $\varrho \in D_n$.

For determination of the constants of integration, $c_0,..., c_n$, one can use the boundary condition, $m_1(1)=0$ and continuity requirements imposed on the bending moment m_1 at $\varrho = \alpha_i$ ($i = 1,..., n$). Thus, according to Eqs. (8), (21) and (22), one has

$$c_j = \sum_{i=j+1}^{n}\left[\frac{\alpha_i^2}{2}\left(\gamma_i^2 - \gamma_{i-1}^2\right) + \frac{\alpha_i^3}{9k}(\gamma_i - \gamma_{i-1})\right] + \frac{1}{2}\left(1 + \frac{q}{k} - \gamma_n^2\right) - \frac{\gamma_n}{9k} \tag{23}$$

for $j = 0,..., n-1$ and

$$c_n = \frac{1}{2}\left(1 + \frac{q}{k} - \gamma_n^2\right) - \frac{\gamma_n}{9k}. \tag{24}$$

Finally, satisfying the boundary requirement at the internal edge, $m_1(\alpha) = \gamma_0^2$, making use of Eqs. (21) and (23), one obtains

$$-\frac{1}{2}\left(\gamma_0^2 + 1 + \frac{q}{k}\right) - \frac{\alpha\gamma_0}{9k}(3\ \ln \alpha - 1) + \frac{\alpha}{3k}\sum_{i=1}^{n}(\gamma_{i-1} - \gamma_i)\ln \alpha_i + \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[\frac{\alpha_i^2}{2}\left(\gamma_i^2 - \gamma_{i-1}^2\right) + \frac{\alpha_i^3}{9k}(\gamma_i - \gamma_{i-1})\right] + \frac{1}{2\alpha^2}\left(1 + \frac{q}{k} - \gamma_n^2\right) - \frac{\gamma_n}{9k\alpha^2} = 0.$$

From Eq. (25), it immediately follows that the load carrying capacity of the shell of piece-wise constant thickness can be presented as

$$q = -k + \frac{k}{1-\alpha^2}\left(\gamma_0^2\alpha^2 + \gamma_n^2\right) + \frac{4\gamma_n}{9(1-\alpha^2)} + \frac{2k\alpha^2}{\alpha^2 - 1} - \left\{-\frac{\alpha\gamma_0}{9k}(3 \ln \alpha - 1) + \frac{\alpha}{3k}\sum_{i=1}^{n}(\gamma_{i-1} - \gamma_i)\ln \alpha_i + \sum_{i=1}^{n}\left[\frac{\alpha_i^2}{2}(\gamma_i - \gamma_{i-1}) + \frac{\alpha_i^3}{9k}(\gamma_i - \gamma_{i-1})\right]\right\} - \frac{2\gamma_n}{9(\alpha^2 - 1)}, \tag{26}$$

provided no plastic hinges occur at the cross-sections, $\varrho = \alpha_i$ ($i = 1,..., n$), where the thickness has jumps.

The stress distribution established above must be statically admissible. Therefore, the solution (Eqs. (14) and (26)) has to satisfy the requirements

$$0 \le n_1 \le \gamma_j$$

and

$$0 \le m_1 \le \gamma_j^2$$

for $\varrho \in D_j$ ($j = 0, \ldots, n$).

Evidently, the most dangereous cross-sections are at $\varrho = \alpha_j$. Thus, one has to check if $n_1(\alpha_j) \le \gamma_j$ and

$$m_1(\alpha_j) \le \gamma_j^2 \tag{27}$$

for $j = 1, \ldots, n$.

Introducing new variables θ_j ($j = 1, \ldots, n$), one can present the inequalities of Eq. (27) in the form of equalities

$$m_1(\alpha_j) - \gamma_j^2 + \theta_j^2 = 0 \tag{28}$$

for $j = 1, \ldots, n$. Making use of Eqs. (22) and (23), one can put Eq. (28) into the form

$$\frac{1}{2}\left(\gamma_n^2 - 1 - \frac{q}{k}\right) - \frac{\gamma_j \alpha_j}{9k}(3 \ln \alpha_j - 1) + \frac{1}{\alpha_j^2} \sum_{i=j+1}^{n} \left[\frac{\alpha_i^2}{2}\left(\gamma_i^2 - \gamma_{i-1}^2\right) + \frac{\alpha_i^3}{9k}(\gamma_i - \gamma_{i-1}) \right] + \frac{1}{2\alpha_j^2}\left(1 + \frac{q}{k} - \gamma_n^2\right) - \frac{1}{9k\alpha_j^2}\gamma_n - \gamma_j^2 + \theta_j^2 = 0$$

for $j = 1, \ldots, n$.

In order to define the minimum of the cost function Eq. (6) subjected to the constraints (Eqs. (26) and (29)), let us introduce an extended Lagrangian function:

$$\begin{aligned} I = {} & \sum_{j=0}^{n} \gamma_j \left(\alpha_{j+1}^2 - \alpha_j^2\right) + \lambda_0 \Bigg\{ -k + \frac{k}{1-\alpha^2} + \frac{4\gamma_n}{9(1-\alpha^2)} + \frac{2k\alpha^2}{\alpha^2 - 1} \Bigg[-\frac{\alpha\gamma_0}{9k}(3 \ln \alpha - 1) \\ & + \frac{3\alpha}{k} \sum_{j=1}^{n} (\gamma_{j-1} - \gamma_j) \ln \alpha_j + \sum_{j=1}^{n} \left[\frac{\alpha_j^2}{2}(\gamma_j - \gamma_{j-1}) + \frac{\alpha_j^3}{9k}(\gamma_j - \gamma_{j-1}) \right] \Bigg] - q \Bigg\} \\ & + \sum_{j=1}^{n} \lambda_j \Bigg\{ \frac{1}{2}\left(\gamma_n^2 - 1 - \frac{q}{k}\right) - \frac{\gamma_j \alpha_j}{9k}(3 \ln \alpha_j - 1) + \frac{1}{2\alpha_j^2}\left(1 + \frac{q}{k} - \gamma_n^2\right) - \frac{\gamma_n}{9k\alpha_j^2} + \frac{1}{\alpha_j^2} \sum_{i=j+1}^{n} \left[\frac{\alpha_i^2}{2}\left(\gamma_i^2 - \gamma_{i-1}^2\right) \right. \\ & \left. + \frac{\alpha_i^3}{9k}(\gamma_i - \gamma_{i-1}) \right] - \gamma_j^2 + \theta_j^2 \Bigg\}, \end{aligned} \tag{30}$$

where $\lambda_0, \ldots, \lambda_n$ stand for Lagrangian multipliers.

The necessary conditions of the constrained minimum of the cost function Eq. (6) may be presented according to Eq. (30) as

$$2\alpha_j(\gamma_{j-1}-\gamma_j)+\frac{2k\alpha^2}{\alpha^2-1}\lambda_0\left[\frac{3\alpha}{k}(\gamma_{j-1}-\gamma_j)\cdot\frac{1}{\alpha_j}+\alpha_j(\gamma_j-\gamma_{j-1})+\frac{\alpha_j^2}{3k}(\gamma_j-\gamma_{j-1})\right]+\lambda_j\left\{-\frac{\gamma_j}{9k}(3\ln\alpha_j-1+3)\right.$$
$$\left.-\frac{1}{\alpha_j^3}\left(1+\frac{q}{k}-\gamma_n^2\right)+\frac{2\gamma_n}{9k\alpha_j^3}-\frac{2}{\alpha_j^3}\sum_{i=j+1}^{n}\left[\frac{\alpha_i^2}{2}(\gamma_i^2-\gamma_{i-1}^2)+\frac{\alpha_i^3}{9k}(\gamma_i-\gamma_{i-1})\right]\right\}=0$$

$$\lambda_0\theta_j=0 \tag{31}$$

for $j = 1,..., n$ and

$$\alpha_{j+1}^2-\alpha_j^2+\frac{2k\alpha^2}{\alpha^2-1}\lambda_0\left[\frac{3\alpha}{k}(-\ln\alpha_j+\ln\alpha_{j+1})+\frac{1}{2}\left(\alpha_j^2-\alpha_{j+1}^2\right)+\frac{1}{9k}\left(\alpha_j^3-\alpha_{j+1}^3\right)\right]\lambda_j\left\{-\frac{\alpha_j}{9k}(3\ln\alpha_j-1)\right.$$
$$\left.+\frac{1}{\alpha_j^2}\left[-\alpha_{j+1}^2\gamma_j-\frac{\alpha_{j+1}^3}{9k}\right]\right\}=0$$

for $j = 1,..., n-1$, as well as

$$\alpha_1^2-\alpha^2+\frac{2\lambda_0k\alpha^2}{\alpha^2-1}\left\{-\frac{\alpha}{9k}(3\ln\alpha-1)+\frac{3\alpha}{k}\ln\alpha_1-\frac{\alpha_1^2}{2}-\frac{\alpha_1^3}{9k}\right\}=0$$

$$1-\alpha_n^2+\lambda_0\left\{\frac{4}{9(1-\alpha^2)}+\frac{2k\alpha^2}{\alpha^2-1}\left[\frac{3\alpha}{k}(-\ln\alpha_n)+\frac{\alpha_n^2}{2}+\frac{\alpha_n^3}{9k}\right]\right\}+\lambda_n\left\{-\gamma_n-\frac{\alpha_n}{9k}(3\ln\alpha_n-1)-\frac{1}{\alpha_n^2}\gamma_n\right.$$
$$\left.-\frac{1}{9k\alpha_n^2}+\frac{1}{\alpha_n^2}\left(\alpha_n^2\gamma_n+\frac{\alpha_n^3}{9k}\right)\right\}=0,$$

corresponding to $j = 0$ and $j=n$, respectively.

It follows from the second equation in Eq. (31) that $\lambda_j=0$, if $\theta_j\neq 0$ and $\theta_j=0$, if $\lambda_j\neq 0$. Thus $\lambda_j=0$ if $m_1(\alpha_j)<\gamma_j^2$. However, it is reasonable to assume that the material of the shell is utilized in the most efficient manner if the cross-sections where the thickness varies rapidly are stressed maximally, e.g. $m_1(\alpha_j)=\gamma_j^2$.

In this case, design parameters can be defined from Eqs. (31)–(33), making use of Eqs. (16) and (19). Note that these systems of equations include $3n + 2$ equations, whereas the number of unknowns $(\lambda_j,\alpha_j,\gamma_j)$ is equal to $3n + 2$, as well.

The set of equations (Eqs. (26) and (29), Eqs. (31)–(33)) has been solved numerically by making use of the Newton–Raphson method.

5. Numerical results

The results of the calculations are presented in Figs. 3–6 and Tables 1–3. The load carrying capacity of a conical shell loaded by the central rigid boss is depicted in Fig. 3. The shell is simply supported at the outer edge, whereas the thickness of the shell is constant. The curves labeled as 1, 2, 3 and 4 correspond to $k = 0.1$; $k = 0.3$; $k = 0.9$ and $k = 1.5$, respectively. It can be seen from Fig. 3 that the greater the parameter k, the greater the limit load corresponding to the same value of internal radius.

On the other hand, for a fixed value of the shell parameter k, the load carrying capacity increases

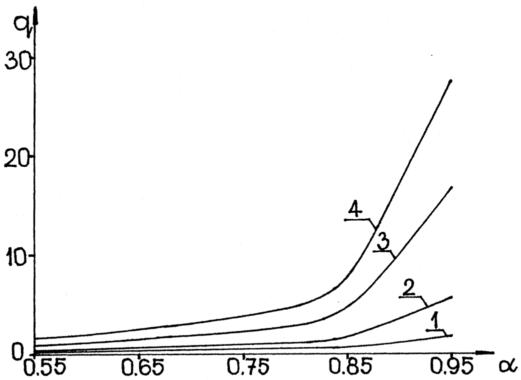

Fig. 3. Load carrying capacity vs. internal radius.

with increasing internal radius of the shell. However, a rapid increase takes place when a/R tends to unity.

Numerical results have been obtained for shells with two and three jumps in the thickness. In Figs. 4 and 5 are presented the distributions of the membrane force and the bending moment, respectively, for the shell with three different thicknesses. The bending moment is computed with respect to the middle surface of the shell. The solid lines in Figs. 4 and 5 correspond to the shell with piece-wise constant thickness, whereas the dashed lines are associated with the reference shell of constant thickness. The curves labeled as 1 and 2 correspond to the cases $k = 0.3$ and $k = 0.9$, respectively, in Figs. 4 and 5. It can be seen from Fig. 4 that the membrane force corresponding to the optimized shell is smaller than that corresponding to the shell of constant thickness.

However, the principal bending moment is smaller in the case of the shell of constant thickness (Fig. 5). The stress distributions depend relatively weakly on the values of the geometrical parameter k.

The economy of the optimal design established could be assessed by the ratio $e = V/V_*$ where V is the optimal material volume of the shell wall. Here, V_* stands for the corresponding material volume of the reference shell of constant thickness. Evidently,

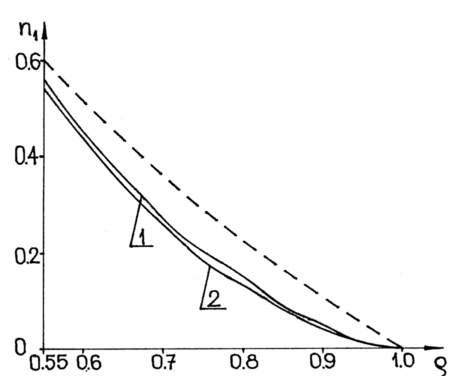

Fig. 4. Membrane force distribution along the shell generator.

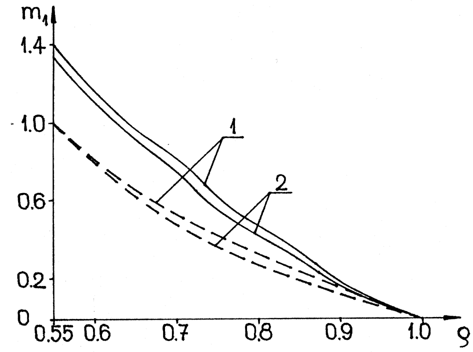

Fig. 5. Bending moment distribution along the shell.

$$V = \frac{\pi h_* R^2}{\cos\varphi}\left[\gamma_0\left(\alpha_1^2 - \alpha^2\right) + \gamma_1\left(\alpha_2^2 - \alpha_1^2\right) + \gamma_2\left(1 - \alpha_2^2\right)\right] \tag{34}$$

in the case of two jumps in the thickness. Similarily, in the case of the reference shell of constant thickness,

$$V_* = \frac{\pi h_* R^2}{\cos\varphi}(1 - \alpha^2). \tag{35}$$

The values of the coefficient of economy, calculated with the aid of Eqs. (34) and (35), are accommodated in the last columns of Tables 1 and 2.

The data presented in Tables 1 and 2 correspond to the shell with two jumps in the thickness. Table 1 corresponds to the shell with $k = 0.3$, whereas Table 2 is associated with the case when $k = 0.9$.

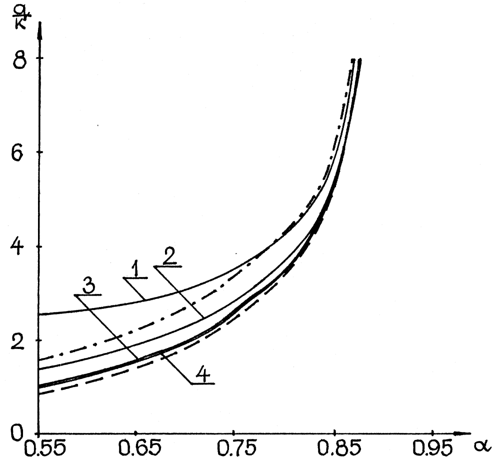

Fig. 6. Lower bound on the limit load as a function of internal radius.

Table 1
Optimal designs for $k = 0.3$

α	q	α_1	α_2	γ_0	γ_1	γ_2	e
0.55	0.431	0.710	0.869	1.178	0.888	0.568	0.860
0.65	0.582	0.778	0.900	1.121	0.850	0.548	0.836
0.75	0.880	0.843	0.929	1.073	0.820	0.533	0.816
0.85	1.631	0.907	0.958	1.036	0.798	0.523	0.802
0.95	5.578	0.969	0.986	1.009	0.785	0.520	0.796

Table 2
Optimal designs for $k = 0.9$

α	q	α_1	α_2	γ_0	γ_1	γ_2	e
0.55	0.951	0.709	0.869	1.162	0.850	0.531	0.828
0.65	1.459	0.777	0.899	1.107	0.825	0.524	0.813
0.75	2.423	0.842	0.929	1.065	0.806	0.520	0.803
0.85	4.756	0.907	0.958	1.033	0.792	0.518	0.797
0.95	16.686	0.969	0.986	1.009	0.785	0.520	0.795

Table 3
Lower-bound solution

α	q/k	α_1	α_2	γ_0	γ_1	γ_2	e
0.55	0.867	0.709	0.869	1.150	0.828	0.510	0.809
0.65	1.463	0.776	0.899	1.098	0.811	0.511	0.801
0.75	2.571	0.842	0.929	1.060	0.798	0.513	0.797
0.85	5.207	0.906	0.958	1.031	0.790	0.516	0.795
0.95	18.513	0.969	0.986	1.009	0.784	0.519	0.771

It can be seen from Tables 1 and 2 that the eventual material saving depends on the radius of the central rigid boss. It is somewhat surprising that the greater the internal radius, the greater the material saving. For instance, in the case that $k = 0.3$ and $a/R = 0.95$, one can save more than 20% of the material when utilizing the shell of piece-wise constant thickness with three different thicknesses. Calculations carried out show that in the case of the shell with two different thicknesses, the eventual material saving is about 14.5%, as shown by the authors (Lellep and Puman, 1994). However, if $a/R = 0.55$, then the corresponding percentages are 14% and 10.4%, respectively.

It appears that a simple lower bound to the load carrying capacity can be obtained when taking $n_1 = n_2 = 0$ and making use of the same yield regime on the plane of bending moments. Corresponding results are presented in Table 3. It can be seen from Tables 1–3 that the values of the design parameters are quite close to each other despite the fact that corresponding solutions have been obtained under different assumptions.

Note that the validity of the exact solution (as well as the approximate one) is restricted by statical constraints $0 \leq n_1 \leq \gamma_j$; $0 \leq m_1 \leq \gamma_j^2$ for $\varrho \in D_j$. Calculations carried out showed that these inequalities are met, if $a > R/2$. However, the lower bound solution is valid for each value of the internal radius a.

The lower bound of the load carrying capacity is presented in Fig. 6 (dashed line). The curves 1, 2, 3 and 4 present the exact solutions in the cases $k = 0.1$; $k = 0.3$; $k = 0.9$ and $k = 1.5$, respectively. It is worthy of note that the curves corresponding to the exact solution tend uniformly to the lower bound solution (dashed line) if the parameter k increases.

The dot–dashed line in Fig. 6 is associated with the lower bound solution in the case of generalized square yield condition. It is somewhat surprising that the results are quite close to each other.

6. Concluding remarks

A minimum weight design technique has been developed for conical shells of piece-wise constant thickness. The shells under consideration are loaded by the rigid central boss, whereas the material of the shell wall is considered to be an ideal rigid-plastic material obeying the generalized diamond yield condition suggested by Jones and Ich (1972). Numerical results have been presented for the shells with 2 and 3 different thicknesses.

A simple lower bound technique was used to predict the limit load and design parameters of the shell. Numerical analysis showed that the results obtained by different methods are surprisingly close to each other.

It was established that the greater part of the eventual material saving can be achieved when using a shell with two different thicknesses (with one step in the thickness). However, when using a shell with two steps in the thickness, remarkable additional saving can be obtained. For instance, in the case $k = 0.3$ and $a = 0.55R$, the design with one step gives an approximately 10% material saving in comparison to the shell of constant thickness. The use of the design with two steps admits to get 4% of additional economy. When adding the number of steps, the increase in the material saving is less remarkable. It is clear that in the limit case when n tends to infinity, the solution for the shell with piece-wise constant thickness tends to that corresponding to the functionally graded thickness. However, the determination of the optimal solution for the shell with continuously varying thickness will be a task for future work.

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